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Introduction to Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems

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Preface

This textbook is an enlarged and enhanced version of the authors' lecture notes used for a graduate course in fuzzy sets, fuzzy logic, fuzzy systems, and fuzzy control theories. This course has been taught for seven years at the University of Houston, with emphasis on fuzzy systems and fuzzy control, regarding both basic mathematical theories and their potential engineering applications.

The word "fuzzy" is perhaps no longer fuzzy to many engineers today. Introduced in the earlier 1970s, fuzzy systems and fuzzy control theories as an emerging technology targeting industrial applications have added a promising new dimension to the existing domain of conventional control systems engineering. It is now a common belief that when a complex physical system does not provide a set of differential or difference equations as a precise or reasonably accurate mathematical model, particularly when the system description requires certain human experience in linguistic terms, fuzzy systems and fuzzy control theories have some salient features and distinguishing merits over many other approaches.

Fuzzy control methods and algorithms, including many specialized software and hardware available on the market today, may be classified as one type of intelligent control. This is because fuzzy systems modeling, analysis, and control incorporate a certain amount of human knowledge into its components (fuzzy sets, fuzzy logic, and fuzzy rule base). Using human expertise in system modeling and controller design is not only advantageous but often necessary. Classical controller design has already incorporated human skills and knowledge: for instance, what type of controller to use and how to determine the controller structure and parameters largely depend on the decision and preference of the designer, especially when multiple choices are possible. The relatively new fuzzy control technology provides one more choice for this consideration; it has the intention to be an alternative, rather than a simple replacement, of the existing control techniques such as classical control and other intelligent control methods (e.g., neural networks, expert systems, etc.). Together, they supply systems and control engineers with a more complete toolbox to deal with the complex, dynamic, and uncertain real world. Fuzzy control technology is one of the many tools in this toolbox that is developed not only for elegant mathematical theories but, more importantly, for many practical problems with various technical challenges.

Compared with conventional approaches, fuzzy control utilizes more information from domain experts and relies less on mathematical modeling about a physical system.

On the one hand, fuzzy control theory can be quite heuristic and somewhat ad hoc. This sometimes is preferable or even desirable, particularly when low-cost and easy operations are required where mathematical rigor is not the main concern. There are many examples of this kind in industrial applications, for which fuzzy sets and fuzzy logic are easy to use. Within this context, determining a fuzzy set or a fuzzy rule base seems to be somewhat subjective, where human knowledge about the underlying physical system comes into play. However, this may not be any more subjective than selecting a suitable mathematical model in the deterministic control approach ("linear or nonlinear?" "if linear, what's the order or dimension and, yet, if nonlinear, what kind of nonlinearity?" "what kind of optimality criterion to use?" "what kind of norm for robustness measure?" etc.). It is also not much more subjective than choosing a suitable distribution function in the stochastic control approach ("Gaussian or non-Gaussian noise?" "white noise or just unknown but bounded uncertainty?" and the like). Although some of these questions can be answered on the basis of statistical analysis of available empirical data in classical control systems, the same is true for establishing an initial fuzzy rule base in fuzzy control systems.

On the other hand, fuzzy control theory can be rigorous and fuzzy controllers can have precise and analytic structures with guaranteed closed-loop system stability and some performance specifications, if such characteristics are intended. In this direction, the ultimate objective of the current fuzzy systems and fuzzy control research is appealing: the fuzzy control system technology is moving toward a solid foundation as part of the modern control theory. The trend of a rigorous approach to fuzzy control, starting from the mid-1980s, has produced many exciting and promising results. For instance, some analytic structures of fuzzy controllers, particularly fuzzy PID controllers, and their relationship with corresponding conventional controllers are much better understood today. Numerous analysis and design methods have been developed, which have turned the earlier "art" of building a working fuzzy controller to the "science" of systematic design. As a consequence, the existing analytical control theory has made the fuzzy control systems practice safer, more efficient, and more cost-effective.

This textbook represents a continuing effort in the pursuit of analytic theory and rigorous design for fuzzy control systems. More specifically, the basic notion of fuzzy mathematics (Zadeh fuzzy set theory, fuzzy membership functions, interval and fuzzy number arithmetic operations) is first studied in this text. Consequently, in a comparison with the classical two-valued logic, the fundamental concept of fuzzy logic is introduced. The ultimate goal of this course is to develop an elementary practical theory for automatic control of uncertain or imperfectly modeled systems encountered in engineering applications using fuzzy mathematics and fuzzy logic, thereby offering an alternative approach to control systems design and analysis under irregular conditions, for which conventional control systems theory may not be able to manage or well perform. Therefore, this part of the text on fuzzy mathematics and fuzzy logic is followed by the basic fuzzy systems theory (Mamdani and Takagi-Sugeno modeling, along with parameter estimation and system identification) and fuzzy control theory. Here, fuzzy control theory is introduced, first based on the developed fuzzy system modeling, along with the concepts of controllability, observability, and stability, and then based on the well-known classical Proportional-Integral-Derivative (PID) controllers theory and design methods. In particular, fuzzy PID controllers are studied in greater detail. These controllers have precise analytic structures, with rigorous analysis and guaranteed closed-loop system stability; they are comparable, and also compatible, with the classical PID controllers. To that end, fuzzy adaptive and optimal control issues are also discussed, albeit only briefly, followed by some potential industrial application examples.

The primary purpose of this course is to provide some rather systematic training for systems and control majors, both senior undergraduate and first-year graduate students, and to familiarize them with some fundamental mathematical theory and design methodology in fuzzy control systems. We have tried to make this book self-contained, so that no preliminary knowledge of fuzzy mathematics and fuzzy control systems theory is needed to understand the material presented in this textbook. Although we assume that the students are aware of the classical set theory, two-valued logic, and elementary classical control systems theory, the fundamentals of these subjects are briefly reviewed throughout for their convenience.

Some familiar terminology in the field of fuzzy control systems has become quite standard today. Therefore, as a textbook written in a classical style, we have taken the liberty to omit some personal and specialized names such as “TS fuzzy model” and “t-norm.” One reason is that too many names have to be given to too many items in doing so. Nevertheless, closely related references are given at the end of each chapter for crediting and for the reader’s further reading. Also, we have indicated by * in the Table of Contents those relatively advanced materials that are beyond the basic scope of the present text; they are used for reader's further studies of the subject.

It is our hope that students will benefit from this textbook in obtaining some relatively comprehensive knowledge about fuzzy control systems theory which, together with their mathematical foundations, can in a way better prepare them for the rapidly developing applied control technologies in modern industry.

The Authors

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CHAPTER 1

Fuzzy Set Theory

The classical set theory is built on the fundamental concept of "set" of which an individual is either a member or not a member. A sharp, crisp, and unambiguous distinction exists between a member and a nonmember for any well-defined "set" of entities in this theory, and there is a very precise and clear boundary to indicate if an entity belongs to the set. In other words, when one asks the question "Is this entity a member of that set?" The answer is either "yes" or "no." This is true for both the deterministic and the stochastic cases. In probability and statistics, one may ask a question like "What is the probability of this entity being a member of that set?" In this case, although an answer could be like "The probability for this entity to be a member of that set is 90%," the final outcome (i.e., conclusion) is still either "it is" or "it is not" a member of the set. The chance for one to make a correct prediction as "it is a member of the set" is 90%, which does not mean that it has 90% membership in the set and in the meantime it possesses 10% non-membership. Namely, in the classical set theory, it is not allowed that an element is in a set and not in the set at the same time. Thus, many real-world application problems cannot be described and handled by the classical set theory, including all those involving elements with only partial membership of a set. On the contrary, fuzzy set theory accepts partial memberships, and, therefore, in a sense generalizes the classical set theory to some extent.

In order to introduce the concept of fuzzy sets, we first review the elementary set theory of classical mathematics. It will be seen that the fuzzy set theory is a very natural extension of the classical set theory, and is also a rigorous mathematical notion.

I. CLASSICAL SET THEORY

A. Fundamental Concepts

Let **S** be a nonempty set, called the *universe set* below, consisting of all the possible elements of concern in a particular context. Each of these elements is called a *member*, or an *element*, of **S**. A union of several (finite or infinite) members of **S** is called a *subset* of **S**. To indicate that a member s of **S** belongs to a subset S of **S**, we write

$$s \in S.$$

If s is not a member of S, we write

$$s \notin S.$$

To indicate that S is a subset of **S**, we write

$$S \subset \mathbf{S}.$$

Usually, this notation implies that S is a strictly proper subset of $\mathbf{S}$ in the sense that there is at least one member $x \in \mathbf{S}$ but $x \notin S$. If it can be either $S \subset \mathbf{S}$ or $S = \mathbf{S}$, we write

$$S \subseteq \mathbf{S}.$$

An empty subset is denoted by $\emptyset$. A subset of certain members that have properties $P_1, \ldots, P_n$ will be denoted by a capital letter, say A, as

$$A = \{\, a \mid a \text{ has properties } P_1, \ldots, P_n \,\}.$$

An important and frequently used universe set is the n-dimensional Euclidean space R^n. A subset $A \subseteq \mathrm{R}^n$ that is said to be *convex* if

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in A \qquad \text{and} \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in A$$

implies

$$\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in A \qquad \text{for any } \lambda \in [0,1].$$

Let A and B be two subsets. If every member of A is also a member of B, i.e., if $a \in A$ implies $a \in B$, then A is said to be a *subset* of B. We write $A \subset B$. If both $A \subset B$ and $B \subset A$ are true, then they are *equal*, for which we write $A = B$. If it can be either $A \subset B$ or $A = B$, then we write $A \subseteq B$. Therefore, $A \subset B$ is equivalent to both $A \subseteq B$ and $A \neq B$.

The *difference* of two subsets A and B is defined by

$$A - B = \{\, c \mid c \in A \text{ and } c \notin B \,\}.$$

In particular, if $A = \mathbf{S}$ is the universe set, then $\mathbf{S} - B$ is called the *complement* of B, and is denoted by $\overline{B}$, i.e.,

$$\overline{B} = \mathbf{S} - B.$$

Obviously,

$$\overline{\overline{B}} = B, \qquad \overline{\mathbf{S}} = \emptyset, \qquad \text{and} \qquad \overline{\emptyset} = \mathbf{S}.$$

Let $r \in \mathrm{R}$ be a real number and A be a subset of R. Then the *multiplication* of r and A is defined to be

$$r\,A = \{\, r\,a \mid a \in A \,\}.$$

The *union* of two subsets A and B is defined by

$$A \cup B = B \cup A = \{\, c \mid c \in A \text{ or } c \in B \,\}.$$

Thus, we always have

$$A \cup \mathbf{S} = \mathbf{S}, \qquad A \cup \emptyset = A, \qquad \text{and} \qquad A \cup \overline{A} = \mathbf{S}.$$

The *intersection* of two subsets A and B is defined by

$$A \cap B = B \cap A = \{\, c \mid c \in A \text{ and } c \in B \,\}.$$

Obviously,

$$A \cap \mathbf{S} = A, \qquad A \cap \emptyset = \emptyset, \qquad \text{and} \qquad A \cap \overline{A} = \emptyset.$$

Two subsets A and B are said to be *disjoint* if

$$A \cap B = \emptyset.$$

Basic properties of the classical set theory are summarized in Table 1.1, where $A \subseteq \mathbf{S}$ and $B \subseteq \mathbf{S}$.

Table 1.1 Properties of Classical Set Operations

Involutive law	$\overline{\overline{A}} = A$
Commutative law	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associative law	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive law	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cup A = A$
	$A \cap A = A$
	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$
	$A \cup (\overline{A} \cap B) = A \cup B$
	$A \cap (\overline{A} \cup B) = A \cap B$
	$A \cup \mathbf{S} = \mathbf{S}$
	$A \cap \varnothing = \varnothing$
	$A \cup \varnothing = A$
	$A \cap \mathbf{S} = A$
	$A \cap \overline{A} = \varnothing$
	$A \cup \overline{A} = \mathbf{S}$
DeMorgan's law	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

In order to simplify the notation throughout the rest of the book, if the universe set **S** has been specified or is not of concern, we simply call any of its subsets a *set*. Thus, we can consider two sets A and B in **S**, and if $A \subset B$ then A is called a *subset* of B.

For any set A, the *characteristic function* of A is defined by

$$\mathbf{X}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

It is easy to verify that for any two sets A and B in the universe set **S** and for any element $x \in \mathbf{S}$, we have

$$\mathbf{X}_{A \cup B}(x) = \max\{\, \mathbf{X}_A(x), \mathbf{X}_B(x) \,\},$$
$$\mathbf{X}_{A \cap B}(x) = \min\{\, \mathbf{X}_A(x), \mathbf{X}_B(x) \,\},$$
$$\mathbf{X}_{\overline{A}}(x) = 1 - \mathbf{X}_A(x).$$

B*. Elementary Measure Theory of Sets

In this subsection, we briefly review the basic notion of *measure* in the classical set theory which, although may not be needed throughout this book, will be useful in further studies of some advanced fuzzy mathematics.

Let **S** be the universe set and **A** a nonempty family of subsets of **S**. Let, moreover,

$$\mu: \quad \mathbf{A} \to [0,\infty]$$

be a nonnegative real-valued function defined on (subsets of) **A**, which may assume the value ∞.

A set B in **A**, denoted as an element of **A** by $B \in \mathbf{A}$, is called a *null set* with respect to μ if $\mu(B) = 0$, where

$$\mu(B) = \{ \mu(b) \mid b \in B \}.$$

μ is said to be *additive* if

$$\mu\Big(\bigcup_{i=1}^{n} A_i \Big) = \bigcup_{i=1}^{n} \mu(A_i)$$

for any finite collection $\{A_1,...,A_n\}$ of sets in **A** satisfying both $\bigcup_{i=1}^{n} A_i \in \mathbf{A}$ and $A_i \cap A_j = \emptyset$, $i \neq j$, $i,j=1,...,n$. μ is said to be *countably additive* if $n = \infty$ in the above. Moreover, μ is said to be *subtractive* if

$$A \in \mathbf{A}, \quad B \in \mathbf{A}, \quad A \subseteq B, \quad B - A \in \mathbf{A}, \quad \text{and} \quad \mu(B) < \infty$$

together imply

$$\mu(B - A) = \mu(B) - \mu(A).$$

It can be verified, however, that if μ is additive then it is also subtractive.

Now, μ is called a *measure* on **A** if it is countably additive and there is a nonempty set $C \in \mathbf{A}$ such that $\mu(C) < \infty$.

For example, if we define a function μ by $\mu(A) = 0$ for all $A \in \mathbf{A}$, then μ is a measure on **A**, which is called the *trivial measure*. As the second example, suppose that **A** contains at least one finite set and define μ by $\mu(A)$ = the number of elements belonging to **A**. Then μ is a measure on **A**, which is called the *natural measure*.

A measure μ on **A** has the following two simple properties: (i) $\mu(\emptyset) = 0$, and (ii) μ is finitely additive.

Let μ be a measure on **A**. Then a set $A \in \mathbf{A}$ is said to have a *finite measure* if $\mu(A) < \infty$, and have a σ-*finite measure* if there is a sequence $\{A_i\}$ of sets in **A** such that

$$A \subseteq \bigcup_{i=1}^{\infty} A_i \qquad \text{and} \qquad \mu(A_i) < \infty \qquad \text{for all } i = 1,2,\cdots.$$

μ is *finite* (resp., σ-*finite*) on **A** if every set in **A** has a finite (resp., σ-finite) measure.

A measure μ on **A** is said to be *complete* if

$$B \in \mathbf{A}, \qquad A \subseteq B, \qquad \text{and} \qquad \mu(B) = 0$$

together imply $\mu(A) = 0$. μ is said to be *monotone* if

$$A \in \mathbf{A}, \qquad B \in \mathbf{A}, \qquad \text{and} \qquad A \subseteq B$$

together imply

$$\mu(A) \leq \mu(B).$$

μ is said to be *subadditive* if

$$\mu(A) \leq \mu(A_1) + \mu(A_2)$$

for any $A, A_1, A_2 \in \mathbf{A}$ with $A = A_1 \cup A_2$. μ is said to be *finitely subadditive* if

$$\mu(A) \le \sum_{i=1}^{n} \mu(A_i)$$

for any finite collection $\{A, A_1, \ldots, A_n\}$ of subsets in **A** satisfying $A = \bigcup_{i=1}^{n} A_i$, and μ is said to be *countably subadditive* if n = ∞ in the above.

It can be shown that if μ is countably subadditive and $\mu(\varnothing) = 0$, then it is also finitely subadditive.

Let $A \in \mathbf{A}$. A measure μ on **A** is said to be *continuous from below* at A if

$$\{A_i\} \subset \mathbf{A}, \qquad A_1 \subseteq A_2 \subseteq \ldots, \qquad \text{and} \qquad \lim_{i \to \infty} A_i = A$$

together imply

$$\lim_{i \to \infty} \mu(A_i) = \mu(A),$$

and μ is said to be *continuous from above* at A if

$$\{A_i\} \subset \mathbf{A}, \qquad A_1 \supseteq A_2 \supseteq \ldots, \qquad \mu(A_1) < \infty, \qquad \text{and} \quad \lim_{i \to \infty} A_i = A$$

together imply

$$\lim_{i \to \infty} \mu(A_i) = \mu(A).$$

μ is continuous from below (resp., above) on **A** if and only if it is continuous from below (resp., above) at every set $A \in \mathbf{A}$, and μ is said to be *continuous* if it is continuous both from below and from above (at A, or on **A**).

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be families of subsets of **A** such that $\mathbf{A}_1 \subseteq \mathbf{A}_2$, and let μ_1 and μ_2 be measures on $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. μ_2 is said to be an *extension* of μ_1 if $\mu_1(A) = \mu_2(A)$ for every $A \in \mathbf{A}_1$.

For example, let $\mathbf{A} = (-\infty, \infty)$, $\mathbf{A}_1 = \{ [a,b) \mid -\infty < a < b < \infty \}$, $\mathbf{A}_2$ = family of all finite, disjoint unions of bounded intervals of the form $[c,d)$, and a measure μ_1 be defined on $\mathbf{A}_1$ by

$$\mu_1([a,b)) = b - a.$$

Then μ_1 is countably additive and so is a finite measure on $\mathbf{A}_1$. This μ_1 can be extended to a finite measure μ_2 on $\mathbf{A}_2$ by defining

$$\mu_2([a,b)) = \mu_1([a,b)) \qquad \text{for all } [a,b) \in \mathbf{A}_1.$$

More generally, if f is a finite, nondecreasing, and left-continuous real-valued function of a real variable, then

$$\mu_f([a,b)) := f(b) - f(a) \qquad \text{for all } [a,b) \in \mathbf{A}_1,$$

defines a finite measure on $\mathbf{A}_1$, and it can be extended to be a finite measure μ_2 on $\mathbf{A}_2$.

II. FUZZY SET THEORY

In Section I.A, we have defined the characteristic function $\mathbf{X}_A$ of a set A by

$$\mathbf{X}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

which is an indicator of members and nonmembers of the crisp set A. In the case that an element has only partial membership of the set, we need to generalize this characteristic function to describe the membership grade of this element in the set: larger values denote higher degrees of the membership.

To give more motivation for this concept of partial membership, let us consider the following examples.

Example 1.1. Let **S** be the set of all human beings, used as the universe set, and let

$$S_f = \{ s \in \mathbf{S} \mid s \text{ is old} \}.$$

Then S_f is a "fuzzy subset" of **S** because the property "old" is not well defined and cannot be precisely measured: given a person who is 40 year old, it is not clear if this person belongs to the set S_f. Thus, to make the subset S_f well-defined, we have to quantify the concept "old," so as to characterize the subset S_f in a precise and rigorous way.

For the time being, let us say, we would like to describe the concept "old" by the curve shown in Figure 1.1(a) using common sense, where the only people who are considered to be "absolutely old" are those 120 years old or older, and the only people who are considered to be "absolutely young" are those newborns. Meanwhile, all the other people are old as well as young, depending on their actual ages. For example, a person 40 years old is considered to be "old" with "degree 0.5" and at the same time also "young" with "degree 0.5" according to the measuring curve that we used. We cannot exclude this person from the set S_f described above, nor include him completely. Thus, the curve that we introduce in Figure 1.1(a) establishes a mathematical measure for the "oldness" of a human being, and hence can be used to define the partial membership of any person relative to the subset S_f described above. The curve shown in Figure 1.1(a), which is indeed a generalization of the classical characteristic function $\mathbf{X}_{S_f}$ (it can be used to conclude a person who either "is" or "is not" a member of the subset S_f), is called a *membership function* associated with the subset S_f.

Of course, one may also use the piecewise linear membership function shown in Figure 1.1(b) to describe the same concept of oldness for the same subset S_f, depending on whichever is more meaningful and more convenient in one's concern, where both are reasonable and acceptable in common sense. The reader may suggest many more good candidates for such a membership function for the subset S_f described above. There is yet no fixed, unique, and universal rule or criterion for selecting a membership function for a particular "fuzzy subset" in general: a correct and good membership function is determined by the user based on his scientific knowledge, working experience, and actual need for the particular application in question. This selection is more or less subjective, but the situation is just like in the classical probability theory and statistics where if one says "we assume that the noise is Gaussian and white," what he uses to start with all the rigorous mathematics is a subjective hypothesis that may not be very true, simply because the noise in question may not be exactly Gaussian and may not be perfectly white. Using the same approach, we can say, "we assume that the membership function that

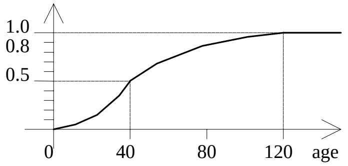

Figure 1.1(a) An example of membership functions.

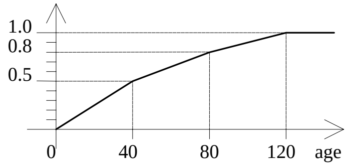

Figure 1.1(b) Another example of membership functions.

describes the oldness is the one given in Figure 1.1(a)," to start with all the rigorous mathematics in the rest of the investigation.

The fuzzy set theory is taking the same logical approach as what people have been doing with the classical set theory: in the classical set theory, as soon as the two-valued characteristic function has been defined and adopted, rigorous mathematics follows; in the fuzzy set case, as soon as a multi-valued characteristic function (the membership function) has been chosen and fixed, a rigorous mathematical theory can be fully developed.

Now, we return to the subset S_f introduced above. Suppose that the membership function associated with it, say the one shown in Figure 1.1(a), has been chosen and fixed. Then, this subset S_f along with the membership function used, which we will denote by $\mu_{S_f}(s)$ with $s \in S_f$, is called a *fuzzy subset* of the universe set **S**. A fuzzy subset thus consists of two components: a subset and a membership function associated with it. This is different from the classical set theory, where all sets and subsets share the same (and the unique) membership function: the two-valued characteristic function mentioned above.

Throughout this book, if no confusion would arise, we will simply call a fuzzy subset a *fuzzy set*, keeping in mind that it has to be a subset of some universe set and has to have a pre-described membership function associated with it.

To familiarize this new concept, let us now discuss one more example.

Example 1.2. Let **S** be the (universe) set of all real numbers, and let

$$S_f = \{\, s \in \mathbf{S} \mid s \text{ is positive and large} \,\}.$$

This subset, S_f, is not well-defined in the classical set theory because, although the statement "s is positive" is precise, the statement "s is large" is vague. However, if we introduce a membership function that is reasonable and meaningful for a particular application for the characterization or measure of the property "large," say the one shown in Figure 1.2 quantified by the function

$$\mu_{S_f}(s) = \begin{cases} 0 & \text{if } s \le 0, \\ 1 - e^{-s} & \text{if } s > 0, \end{cases}$$

then the fuzzy subset S_f, associated with this membership function $\mu_{S_f}(s)$, is well defined.

Similarly, a membership function for the subset