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Introduction to higher order categorical logic

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Introduction to

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Preface

This book makes an effort to reconcile two different attempts to come to grips with the foundations of mathematics. One is mathematical logic, which traditionally consists of proof theory, model theory and the theory of recursive functions; the other is category theory. It has been our experience that, when lecturing on the applications of logic to category theory, we met with approval from logicians and with disapproval from categorists, while the opposite was the case when we mentioned applications of category theory to logic. Unfortunately, to show that the logicians' viewpoint is essentially equivalent to the categorists' one, we have to slightly distort both. For example, categorists may be unhappy when we treat categories as special kinds of deductive systems and logicians may be unhappy when we insist that deductive systems need not be freely generated from axioms and rules of inference. The situation becomes even worse when we take the point of view of universal algebra. For example, combinatory logics are for us certain kinds of algebras, which goes against the grain for those logicians who have spent a life-time studying what we call the free such algebra. On the other hand, cartesian closed categories and even toposes are for us also certain kinds of algebras, although not over sets but over graphs, and this goes against the grain of those categorists who like to think of products and the like as being given only up to isomorphism. To make matters worse, universal algebraists may not be happy when we stress the logical or the categorical aspects, and even graph theorists may feel offended because we have had to choose a definition of graph which is by no means standard.

This is not the first book on categorical logic, as there already exists a classical monograph on first order categorical logic by Makkai and Reyes, not to mention a book on toposes written by a categorist (Johnstone) and a book on topoi written by a logician (Goldblatt), both of whom mention the internal language of toposes*. Our point is rather this: logicians have made

* Let us also draw attention to the important recent book by Barr and Wells, which manages to cover an amazing amount of material without explicit use of logical tools, relying on embedding theorems instead.

three attempts to formulate higher order logic, in increasing power: typed λ-calculus, Martin–Löf type theory and the usual (let us say intuitionistic) type theory. Categorists quite independently, though later, have developed cartesian closed categories, locally cartesian closed categories and toposes. We claim here that typed λ-calculi and cartesian closed categories are essentially the same, in the sense that there is an equivalence of categories (even untyped λ-calculi are essentially the same as certain algebras we call C-monoids). All this will be found in Part I. We also claim that intuitionistic type theories and toposes are closely related, in as much as there is a pair of adjoint functors between their respective categories. This is worked out in Part II. The relationship between Martin–Löf type theories and locally cartesian closed categories was established too recently (by Robert Seely) to be treated here. Logicians will find applications of proof theory in Part I, while many possible applications of proof theory in Part II have been replaced by categorical techniques. They will find some mention of model theory in Part I and more in Part II, but with emphasis on a categorical presentation: models are functors. All discussion of recursive functions is relegated to Part III.

We deliberately excluded certain topics from consideration, such as geometric logic and geometric morphisms. There are other topics which we omitted with some regret, because of limitations of time and space. These include the results of Robert Seely already mentioned, Gödel's *Dialectica interpretation* (1958), which greatly influenced much of this book, the relation between Gödel's double negation translation and double negation sheaves noted by Peter Freyd, Joyal's proof of Brouwer's principle that arrows from R to R in the free topos necessarily represent continuous functions (and related results), the proof that N is projective in the free topos and the important work on graphical algebras by Burroni.

Of course, like other authors, we have some axes to grind. Aside from what some people may consider to be undue emphasis on category theory, logic, universal algebra or graph theory, we stress the following views:

> We decry overzealous applications of Occam's razor.
>
> We believe that type theory is the proper foundation for mathematics.
>
> We believe that the free topos, constructed linguistically but determined uniquely (up to isomorphism) by its universal property, is an acceptable universe of mathematics for a moderate intuitionist and, therefore, that Platonism, formalism and intuitionism are reconcilable philosophies of mathematics.

This may be the place for discussing very briefly who did what. Many results in categorical logic were in the air and were discovered by a number of people simultaneously. Many results were discussed at the Séminaire Bénabou in Paris and published only in preprint form if at all. (Since we are referring to a number of preprints in our bibliography, we should point out that preliminary versions of portions of this book had also been circulated in preprint form, namely Part I in 1982, Part II in 1983 and Part 0 in 1983.) If we are allowed to say to whom we owe the principal ideas exposed in this monograph, we single out Bill Lawvere, Peter Freyd, André Joyal and Dana Scott, and hope that no one whose name has been omitted will be offended.

Let us also take this opportunity to thank all those who have provided us with some feedback on preliminary versions of Parts 0 and I. Again, hoping not to give offence to others, we single out for special thanks (in alphabetic order) Alan Adamson, Bill Anglin, John Gray, Bill Hatcher, Denis Higgs, Bill Lawvere, Fred Linton, Adam Obtułowicz and Birge Zimmermann-Huysgen. We also thank Peter Johnstone for his astute comments on our seminar presentation of Part II. Of course, we take full responsibility for all errors and oversights that still remain.

Finally let us thank Marcia Rodríguez for her conscientious handling of the bibliography, Pat Ferguson for her excellent and patient typing of successive versions of our manuscript and David Tranah for initiating the whole project.

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Montreal, July, 1984

This reprint differs from the original only in the correction of some typographical errors. July 1987

In this reprinting we have repaired various minor misprints and errata. We especially thank Johan van Bentham, Kosta Došen, and Makoto Tatsuta for their careful reading of the text.

Since this book was first published, there has been a tremendous increase of interest in categorical logic among theoretical computer scientists. Of particular importance has been the development of higher-order (= polymorphic) lambda calculi (see Girard's thesis). In the terminology of Part I of this book, such calculi correspond to the deductive systems associated with the intuitionistic type theories of Part II (cf. also R.A.G. Seely, *J. Symb. Logic* **52** (1987), pp. 969–989).

The equational treatment of weak natural numbers objects in Part I has been extended to strong natural numbers objects (see J. Lambek, *Springer LNM* **1348** (1988) 221–229).

Montreal, March, 1994.

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Introduction to category theory

Introduction to Part 0

In Part 0 we recall the basic background in category theory which may be required in later portions of this book. The reader who is familiar with category theory should certainly skip Part 0, but even the reader who is not is advised to consult it only in addition to standard texts.

Most of the material in Part 0 is standard and may also be found in other books. Therefore, on the whole we shall refrain from making historical remarks. However, our exposition differs from treatments elsewhere in several respects.

Firstly, our exposition is slanted towards readers with some acquaintance with logic. Quite early we introduce the notion of a 'deductive system'. For us, this is just a category without the usual equations between arrows. In particular, we do not insist that a deductive system is freely generated from certain axioms, as is customary in logic. In fact, we really believe that logicians should turn attention to categories, which are deductive systems with suitable equations between proofs.

Secondly, we have summarized some of the main thrusts of category theory in the form of succinct slogans. Most of these are due to Bill Lawvere (whose influence on the development of category theory is difficult to overestimate), even if we do not use his exact words. Slogan V represents the point of view of a series of papers by one of the authors in collaboration with Basil Rattray.

Thirdly, we have emphasized the algebraic or equational nature of many of the systems studied in category theory. Just as groups or rings are algebraic over sets, it has been known for a long time that categories with finite products are equational over graphs. More recently, Albert Burroni made the surprising discovery that categories with equalizers are also algebraic over graphs. We have included this result, without going into his more technical concept of 'graphical algebra'.

In Part 0, as in the rest of this book, we have been rather cavalier about set theoretical foundations. Essentially, we are using Gödel–Bernays, as do most mathematicians, but occasionally we refer to universes in the sense of Grothendieck. The reason for our lack of enthusiasm in presenting the foundations properly is our belief that mathematics should be based on a version of type theory, a variant of which adequate for arithmetic and analysis is developed in Part II. For a detailed discussion of these foundational questions see Hatcher (1982, Chapter 8.)

1 Categories and functors

In this section we present what our reader is expected to know about category theory. We begin with a rather informal definition.

Definition 1.1. A *concrete category* is a collection of two kinds of entities, called *objects* and *morphisms*. The former are sets which are endowed with some kind of structure, and the latter are mappings, that is, functions from one object to another, in some sense preserving that structure. Among the morphisms, there is attached to each object A the *identity mapping* 1_A: $A \to A$ such that $1_A(a) = a$ for all $a \in A$. Moreover, morphisms $f: A \to B$ and $g: B \to C$ may be *composed* to produce a morphism $gf: A \to C$ such that $(gf)(a) = g(f(a))$ for all $a \in A$. (See also Exercise 2 below.)

Examples of concrete categories abound in mathematics; here are just three:

Example C1. The category of *sets*. Its objects are arbitrary sets and its morphisms are arbitrary mappings. We call this category '**Sets**'.

Example C2. The category of *monoids*. Its objects are monoids, that is, semigroups with unity element, and its morphisms are homomorphisms, that is, mappings which preserve multiplication (the semigroup operation) and the unity element.

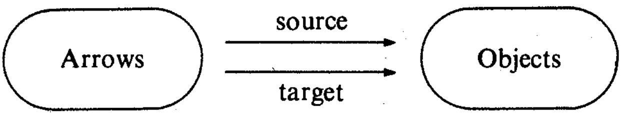
Example C3. The category of *preordered sets*. Its objects are preordered sets, that is, sets with a transitive and reflexive relation on them, and its morphisms are monotone mappings, that is, mappings which preserve this relation.

The reader will be able to think of many other examples: the categories of rings, topological spaces and Banach algebras, to name just a few. In fact, one is tempted to make a generalization, which may be summed up as follows, provided we understand 'object' to mean 'structured set'.

Slogan I. Many objects of interest in mathematics congregate in concrete categories.

We shall now progress from concrete categories to abstract ones, in three easy stages.

Definition 1.2. A *graph* (usually called a *directed graph*) consists of two classes: the class of *arrows* (or *oriented edges*) and the class of *objects* (usually called *nodes* or *vertices*) and two mappings from the class of arrows to the class of objects, called *source* and *target* (often also *domain* and *codomain*).



One writes '$f: A \to B$' for 'source $f = A$ and target $f = B$'. A graph is said to be *small* if the classes of objects and arrows are sets.

Example C4. The category of small *graphs* is another concrete category. Its objects are small graphs and its morphisms are functions F which send arrows to arrows and vertices to vertices so that, whenever $f: A \to B$, then $F(f): F(A) \to F(B)$.

A *deductive system* is a graph in which to each object A there is associated an arrow $1_A: A \to A$, the *identity* arrow, and to each pair of arrows $f: A \to B$ and $g: B \to C$ there is associated an arrow $gf: A \to C$, the *composition* of f with g. A logician may think of the objects as *formulas* and of the arrows as *deductions* or *proofs*, hence of

$$\frac{f: A \to B \quad g: B \to C}{gf: A \to C}$$

as a *rule of inference*. (Deductive systems will be discussed further in Part I.)

A *category* is a deductive system in which the following equations hold, for all $f: A \to B$, $g: B \to C$ and $h: C \to D$:

$$f 1_A = f = 1_B f, \quad (hg)f = h(gf).$$

Of course, all concrete categories are categories. A category is said to be *small* if the classes of arrows and objects are sets. While the concrete categories described in examples 1 to 4 are not small, a somewhat surprising observation is summarized as follows:

Slogan II. Many objects of interest to mathematicians are themselves small categories.

Example C1′. Any set can be viewed as a category: a small *discrete* category. The objects are its elements and there are no arrows except the obligatory identity arrows.

Example C2'. Any monoid can be viewed as a category. There is only one object, which may remain nameless, and the arrows of the monoid are its elements. In particular, the identity arrow is the unity element. Composition is the binary operation of the monoid.

Example C3'. Any preordered set can be viewed as a category. The objects are its elements and, for any pair of objects (a, b), there is at most one arrow $a \to b$, exactly one when $a \leqslant b$.

It follows from slogans I and II that small categories themselves should be the objects of a category worthy of study.

Example C5. The category **Cat** has as objects small categories and as morphisms functors, which we shall now define.

Definition 1.3. A *functor* $F: \mathscr{A} \to \mathscr{B}$ is first of all a morphism of graphs (see Example C4), that is, it sends objects of $\mathscr{A}$ to objects of $\mathscr{B}$ and arrows of $\mathscr{A}$ to arrows of $\mathscr{B}$ such that, if $f: A \to A'$, then $F(f): F(A) \to F(A')$. Moreover, a functor preserves identities and composition; thus

$$F(1_A) = 1_{F(A)}, \quad F(gf) = F(g)F(f).$$

In particular, the identity functor $1_{\mathscr{A}}: \mathscr{A} \to \mathscr{A}$ leaves objects and arrows unchanged and the composition of functors $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{B} \to \mathscr{C}$ is given by

$$(GF)(A) = G(F(A)), \quad (GF)(f) = G(F(f)),$$

for all objects A of $\mathscr{A}$ and all arrows $f: A \to A'$ in $\mathscr{A}$.

The reader will now easily check the following assertion.

Proposition 1.4. When sets, monoids and preordered sets are regarded as small categories, the morphisms between them are the same as the functors between them.

The above definition of a functor $F: \mathscr{A} \to \mathscr{B}$ applies equally well when $\mathscr{A}$ and $\mathscr{B}$ are not necessarily small, provided we allow mappings between classes. Of special interest is the situation when $\mathscr{B} = \mathbf{Sets}$ and $\mathscr{A}$ is small.

Slogan III. Many objects of interest to mathematicians may be viewed as functors from small categories to **Sets**.

Example F1. A set may be viewed as a functor from a discrete one-object category to **Sets**.

Example F2. A small graph may be viewed as a functor from the small category $\cdot \rightrightarrows \cdot$ (with identity arrows not shown) to **Sets**.

Example F3. If $\mathscr{M} = (M, 1, \cdot)$ is a monoid viewed as a one-object category, an $\mathscr{M}$-set may be regarded as a functor from $\mathscr{M}$ to **Sets**. (An *$\mathscr{M}$-set* is a set A together with a mapping $M \times A \to A$, usually denoted by $(m, a) \mapsto ma$, such that $1a = a$ and $(m \cdot m')a = m(m'a)$ for all $a \in A$, m and $m' \in M$.)

Once we admit that functors $\mathscr{A} \to \mathscr{B}$ are interesting objects to study, we should see in them the objects of yet another category. We shall study such functor categories in the next section. For the present, let us mention two other ways of forming new categories from old.

Example C6. From any category (or graph) $\mathscr{A}$ one forms a new category (respectively graph) $\mathscr{A}^{\mathrm{op}}$ with the same objects but with arrows reversed, that is, with the two mappings 'source' and 'target' interchanged. $\mathscr{A}^{\mathrm{op}}$ is called the *opposite* or *dual* of $\mathscr{A}$. A functor from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}$ is often called a *contravariant* functor from $\mathscr{A}$ to $\mathscr{B}$, but we shall avoid this terminology except for occasional emphasis.

Example C7. Given two categories $\mathscr{A}$ and $\mathscr{B}$, one forms a new category $\mathscr{A} \times \mathscr{B}$ whose objects are pairs (A, B), A in $\mathscr{A}$ and B in $\mathscr{B}$, and whose arrows are pairs $(f, g): (A, B) \to (A', B')$, where $f: A \to A'$ in $\mathscr{A}$ and $g: B \to B'$ in $\mathscr{B}$. Composition of arrows is defined componentwise.

Definition 1.5. An arrow $f: A \to B$ in a category is called an *isomorphism* if there is an arrow $g: B \to A$ such that $gf = 1_A$ and $fg = 1_B$. One writes $A \cong B$ to mean that such an isomorphism exists and says that A is *isomorphic* with B.

In particular, a functor $F: \mathscr{A} \to \mathscr{B}$ between two categories is an isomorphism if there is a functor $G: \mathscr{B} \to \mathscr{A}$ such that $GF = 1_{\mathscr{A}}$ and $FG = 1_{\mathscr{B}}$. We also remark that a group is a one-object category in which all arrows are isomorphisms.

To end this section, we shall record three basic isomorphisms. Here **1** is the category with one object and one arrow.

Proposition 1.6. For any categories $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$,

$$\mathscr{A} \times \mathbf{1} \cong \mathscr{A}, \quad (\mathscr{A} \times \mathscr{B}) \times \mathscr{C} \cong \mathscr{A} \times (\mathscr{B} \times \mathscr{C}), \quad \mathscr{A} \times \mathscr{B} \cong \mathscr{B} \times \mathscr{A}.$$

Exercises

1. Prove Propositions 1.4 and 1.6.

2. Show that for any concrete category $\mathscr{A}$ there is a functor $U: \mathscr{A} \to$ **Sets**

which 'forgets' the structure, often called the *forgetful* functor. Clearly U is *faithful* in the sense that, for all $f, g: A \rightrightarrows B$, if $U(f) = U(g)$ then $f = g$. (A more formal version of Definition 1.1 describes a *concrete* category as a pair $(\mathscr{A}, U)$, where $\mathscr{A}$ is a category and $U: \mathscr{A} \to \mathbf{Sets}$ is a faithful functor.)

3. Show that for any category $\mathscr{A}$ there are functors $\Delta: \mathscr{A} \to \mathscr{A} \times \mathscr{A}$ and $\bigcirc_{\mathscr{A}}: \mathscr{A} \to \mathbf{1}$ given on objects A of $\mathscr{A}$ by $\Delta(A) = (A, A)$ and $\bigcirc_{\mathscr{A}}(A) =$ the object of $\mathbf{1}$.

2 Natural transformations

In this section we shall investigate morphisms between functors.

Definition 2.1. Given functors $F, G: \mathscr{A} \rightrightarrows \mathscr{B}$, a *natural transformation* $t: F \to G$ is a family of arrows $t(A): F(A) \to G(A)$ in $\mathscr{B}$, one arrow for each object A of $\mathscr{A}$, such that the following square commutes for all arrows $f: A \to B$ in $\mathscr{A}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{\;t(A)\;} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\;t(B)\;]{} & G(B) \end{array}$$

that is to say, such that

$$G(f)t(A) = t(B)F(f).$$

It is this concept about which it has been said that it necessitated the invention of category theory. We shall give examples of natural transformations later. For the moment, we are interested in another example of a category.

Example C8. Given categories $\mathscr{A}$ and $\mathscr{B}$, the *functor category* $\mathscr{B}^{\mathscr{A}}$ has as objects functors $F: \mathscr{A} \to \mathscr{B}$ and as arrows natural transformations. The *identity* natural transformation $1_F: F \to F$ is of course given by stipulating that $1_F(A) = 1_{F(A)}$ for each object A of $\mathscr{A}$. If $t: F \to G$ and $u: G \to H$ are natural transformations, their *composition* $u \circ t$ is given by stipulating that $(u \circ t)(A) = u(A)t(A)$ for each object A of $\mathscr{A}$.

To appreciate the usefulness of natural transformations, the reader should prove for himself the following, which supports Slogan III.

Proposition 2.2. When objects such as sets, small graphs and $\mathscr{M}$-sets are viewed as functors into **Sets** (see Examples F1 to F3 in Section 1), the morphisms between two objects are precisely the natural transformations. Thus, the categories of sets, small graphs and $\mathcal{M}$-sets may be identified with the functor categories $\mathbf{Sets}^{\mathbf{1}}$, $\mathbf{Sets}^{\cdot\rightrightarrows}$ and $\mathbf{Sets}^{\mathcal{M}}$respectively.

Of course, morphisms between sets are mappings, morphisms between graphs were described in Definition 1.3 and morphisms between $\mathcal{M}$-sets are $\mathcal{M}$-homomorphisms. (An *$\mathcal{M}$-homomorphism* $f: A \to B$ between $\mathcal{M}$-sets is a mapping such that $f(ma) = mf(a)$ for all $m \in M$ and $a \in A$.)

We record three more basic isomorphisms in the spirit of Proposition 1.6.

Proposition 2.3. For any categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$,

$$\mathcal{A}^{\mathbf{1}} \cong \mathcal{A}, \quad \mathcal{C}^{\mathcal{A} \times \mathcal{B}} \cong (\mathcal{C}^{\mathcal{B}})^{\mathcal{A}}, \quad (\mathcal{A} \times \mathcal{B})^{\mathcal{C}} \cong \mathcal{A}^{\mathcal{C}} \times \mathcal{B}^{\mathcal{C}}.$$

We shall leave the lengthy proof of this to the reader. We only mention here the functor $\mathcal{C}^{\mathcal{A} \times \mathcal{B}} \to (\mathcal{C}^{\mathcal{B}})^{\mathcal{A}}$, which will be used later. We describe its action on objects by stipulating that it assigns to a functor $F: \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ the functor $F^*: \mathcal{A} \to \mathcal{C}^{\mathcal{B}}$ which is defined as follows:

For any object A of $\mathcal{A}$, the functor $F^*(A): \mathcal{B} \to \mathcal{C}$ is given by $F^*(A)(B) = F(A, B)$ and $F^*(A)(g) = F(1_A, g)$, for any object B of $\mathcal{B}$ and any arrow $g: B \to B'$ in $\mathcal{B}$.

For any arrow $f: A \to A'$, $F^*(f): F^*(A) \to F^*(A')$ is the natural transformation given by $F^*(f)(B) = F(f, 1_B)$, for all objects B of $\mathcal{B}$.

Finally, to any natural transformation $t: F \to G$ between functors F, G: $\mathcal{A} \times \mathcal{B} \rightrightarrows \mathcal{C}$ we assign the natural transformation $t^*: F^* \to G^*$ which is given by $t^*(A)(B) = t(A, B)$ for all objects A of $\mathcal{A}$ and B of $\mathcal{B}$.

This may be as good a place as any to mention that natural transformations may also be composed with functors.

Definition 2.4. In the situation

$$\mathcal{D} \xrightarrow{L} \mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B} \xrightarrow{K} \mathcal{C},$$

if $t: F \to G$ is a natural transformation, one obtains natural transformations $Kt: KF \to KG$ between functors from $\mathcal{A}$ to $\mathcal{C}$ and $tL: FL \to GL$ between functors from $\mathcal{D}$ to $\mathcal{B}$ defined as follows:

$$(Kt)(A) = K(t(A)), \quad (tL)(D) = t(L(D)),$$

for all objects A of $\mathcal{A}$ and D of $\mathcal{D}$.

If $H: \mathcal{A} \to \mathcal{B}$ is another functor and $u: G \to H$ another natural transformation, then the reader will easily check the following 'distributive laws':

$$K(u \circ t) = (Ku) \circ (Kt), \quad (u \circ t)L = (uL) \circ (tL).$$

If we compare Slogans I and III, we are led to ask: which categories may be viewed as categories of functors into **Sets**? In preparation for an answer to that question we need another definition.

Definition 2.5. If A and B are objects of a category $\mathscr{A}$, we denote by $\mathrm{Hom}_{\mathscr{A}}(A, B)$ the class of arrows $A \to B$. (Later, the subscript $\mathscr{A}$ will often be omitted.) If it so happens that $\mathrm{Hom}_{\mathscr{A}}(A, B)$ is a set for all objects A and B, $\mathscr{A}$ is said to be *locally small.*

One purpose of this definition is to describe the following functor.

Example F4. If $\mathscr{A}$ is a locally small category, then there is a functor $\mathrm{Hom}_{\mathscr{A}}: \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathrm{Sets}$. For an object (A, B) of $\mathscr{A}^{\mathrm{op}} \times \mathscr{A}$, the value of this functor is $\mathrm{Hom}_{\mathscr{A}}(A, B)$, as suggested by the notation. For an arrow $(g, h): (A, B) \to (A', B')$ of $\mathscr{A}^{\mathrm{op}} \times \mathscr{A}$, where $g: A' \to A$ and $h: B \to B'$ in $\mathscr{A}$, $\mathrm{Hom}_{\mathscr{A}}(g, h)$ sends $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ to $hfg \in \mathrm{Hom}_{\mathscr{A}}(A', B')$.

Applying the isomorphism $\mathrm{Sets}^{\mathscr{A}^{\mathrm{op}} \times \mathscr{A}} \to (\mathrm{Sets}^{\mathscr{A}})^{\mathscr{A}^{\mathrm{op}}}$ of Proposition 2.3, we obtain a functor $\mathrm{Hom}^*_{\mathscr{A}}: \mathscr{A}^{\mathrm{op}} \to \mathrm{Sets}^{\mathscr{A}}$ and, dually, a functor $\mathrm{Hom}^*_{\mathscr{A}^{\mathrm{op}}}: \mathscr{A} \to \mathrm{Set}^{\mathscr{A}^{\mathrm{op}}}$. We shall see that the latter functor allows us to assert that $\mathscr{A}$ is isomorphic to a 'full' subcategory of $\mathrm{Sets}^{\mathscr{A}^{\mathrm{op}}}$.

Definition 2.6. A *subcategory* $\mathscr{C}$ of a category $\mathscr{B}$ is any category whose class of objects and arrows is contained in the class of objects and arrows of $\mathscr{C}$ respectively and which is closed under the 'operations' source, target, identity and composition. By saying that a subcategory $\mathscr{C}$ of $\mathscr{B}$ is *full* we mean that, for any objects C, C' of $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(C, C') = \mathrm{Hom}_{\mathscr{B}}(C, C')$.

For example, a proper subgroup of a group is a subcategory which is not full, but the category of Abelian groups is a full subcategory of the category of all groups.

The arrows $F \to G$ in $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$ are natural transformations. We therefore write $\mathrm{Nat}(F, G)$ in place of $\mathrm{Hom}(F, G)$ in $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$.

Objects of the latter category are sometimes called 'contravariant' functors from $\mathscr{A}$ to **Sets**. Among them is the functor $h_A \equiv \mathrm{Hom}_{\mathscr{A}}(-, A)$ which sends the object A' of $\mathscr{A}$ onto the set $\mathrm{Hom}_{\mathscr{A}}(A', A)$ and the arrow $f: A' \to A''$ onto the mapping $\mathrm{Hom}_{\mathscr{A}}(f, 1_A): \mathrm{Hom}_{\mathscr{A}}(A'', A) \to \mathrm{Hom}_{\mathscr{A}}(A', A)$.

The following is known as Yoneda's Lemma.

Proposition 2.7. If $\mathscr{A}$ is locally small and $F: \mathscr{A}^{\mathrm{op}} \to \mathbf{Sets}$, then $\mathrm{Nat}(h_A, F)$ is in one-to-one correspondence with $F(A)$.

Proof. If $a \in F(A)$, we obtain a natural transformation $\check{a}: h_A \to F$ by stipulat-