Abstract Homotopy and
Simple Homotopy Theory
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Abstract Homotopy
and
Simple Homotopy Theory

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This book developed from an initial idea of providing a treatment of abstract homotopy theory as an underlying foundation for an abstract simple homotopy theory. Early on we realised that much of the basic theory was not well known and although often of a quite elementary nature, with much geometric intuition to provide a motivation, it was somehow thought of as 'deep' and 'hard' and perhaps 'esoteric' as well. Perhaps therefore a wider based 'elementary' text was more appropriate, introducing the main features of abstract homotopy theory – but then what were these 'main features'. Different authors had decided on different features. They even seem to disagree as to exactly what constitutes abstract homotopy theory. Our aim in the book that resulted from this idea, is not to give a definite answer to this problem. Abstract homotopy is a very rich and fruitful area of mathematics. It has very many interacting strands or themes running through it, so there was no way that we could provide a thorough treatment of it all. We have therefore selected themes. Themes that reflect our own uses and abuses of this theory, but also that represent and link the different traditions of the subject. We have no 'magic formula' set of axioms to fence in abstract homotopy theory, so to unify those themes we have used some almost philosophic, certainly pedagogic 'aims and objectives' to link them.

First the themes: homotopy theory grew out of paths and cylinders so the use of cylinders and the dual notion of cocylinder should be of first importance. Where possible, as much as possible of the other structure of the abstract homotopy theory should be able to be derived from the structure of the cylinder – so the use of a generating cylinder recurs as a theme. This links in with less cylinder-based approaches such as that of Quillen in which properties of certain classes of maps
are abstracted rather than the cylinder structure and thus a primitive notion of homotopy.

When one looks at the structure of arguments in topological homotopy theory, special structure of the unit interval, unit square etc., is used constantly. In an abstract setting, one way to specify similar structure is via Kan filling or extension conditions. The encoding of structure in terms of filling conditions raises the question: for a given result what filling conditions are needed? This gives a second theme.

A frequently used result in elementary homotopy is Dold's theorem. This and variants such as the 'relativity principle' provide a measure of the strength of an abstract theory of homotopy. This recurs again and again throughout the book.

The use of filling conditions involves the concept of higher homotopies and thus naturally of homotopy coherence. We gradually make this theme more explicit, but within the space available can only scratch the surface of the area. This theme also demands 'explicit' constructions of homotopies. The explicitness also comes in in the final theme of simple homotopy. Simple homotopy is a constructive form of homotopy theory based on constructing homotopy equivalences rather than homotopies between maps.

Turning now to our aims, we could not claim to be exhaustive as to our treatment of subjects so we have tried to design the book to allow easy 'entry' to original source material, via motivating examples and discussions of such material 'in context'. We thus see the role of the graduate textbook as a key to many doors rather than the all embracing tome. A book that 'does everything' does not help the reader to learn, so once started on any subject and having laid a firm foundation, routine verifications are left to the reader. We feel that abstraction without examples is hard, so Chapter III looks at some case studies of abstract homotopy theories.

These are chosen for their usefulness and external interest, not because they fit the theory exactly. What they do is illustrate the problems that arise when attempting to encode or model such a rich set of examples. In the development of the examples the reader is expected
to do some background development work – all will not be laid out on a plate.

The book is thus designed to allow entry into a beautifully rich area which can be loosely called abstract homotopy theory. It can also provide a non-conventional approach to ordinary homotopy theory as we feel it makes explicit parts of that theory that are obscured by the particularities in the topological setting. We have tried to make it reasonably accessible to a beginning graduate student and to make it enjoyable. We hope you will find it so.

Acknowledgements. We would like to thank Kornelia Topp for her constant good humour, excellent typing and \TeX{}ing and Thomas Müller whose \TeX{}pertise in producing diagrams is evident throughout the pages of this book.

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Abstract Homotopy Theory

1. Why Abstract Homotopy Theory?

The history of homotopy theory started towards the end of the 19th century with the study of integrals along paths. Versions of Cauchy’s integral formula and the development of analytic continuation made a knowledge of the behaviour of paths in a space a useful facet to develop. From paths on surfaces to paths in higher dimensional spaces the process of abstraction continued. Paths in high dimensional phase spaces might describe the way in which the many attributes of a moving or changing object might evolve.

Poincaré started exploring the algebra of paths and developed the start of the theory of the fundamental group and groupoid. Homotopy theory and its ‘cousin’ homology theory continued to develop until in the 1950’s homology theory was codified by Eilenberg and Steenrod (1952) and homological algebra by Cartan and Eilenberg (1956). This abstraction of what homology was did not stifle the subject. It provided a unification of ideas so that the abstract proof of a result in homological algebra for one application could provide the necessary key for a breakthrough in a completely different application. Grothendieck’s early work known as the Tôhoku paper (1957), was inspired by problems in algebraic geometry but has been used in most parts of homological algebra and way beyond.

In the topological setting the abstraction of axioms for a homology theory quickly lead to results on ‘extraordinary’ homology and cohomology and a new branch of algebraic topology was formed.

The unification through abstraction leads often to a greater applicability and transparency of proofs giving ‘transportability’ to other
contexts as a bonus. If, as is nowadays common, the abstraction involves a categorical approach, then as a consequence duality enters in a simple manner – you get two proofs for the price of one.

Whilst homology and homological algebra abstracted and prospered the abstraction of homotopy theory was less definite or final, perhaps due to an inherent complexity beyond that of homology. Late in the 1960's Quillen wrote papers using his codification of 'homotopical algebra' (1967), but although extremely useful it does not give the definitive theory. Kan's development of simplicial theory (1957) again provided a rich tool kit which still is used today, but not all homotopy seems to be subsumed in that setting. More recently Baues (1989) has abstracted aspects of homotopy theory to help push through J.H.C. Whitehead's algebraic homotopy program (1950 a) and Heller (1988) has asked what the 'global' picture of homotopy theory should be. (Incidently or accidently Heller asks the same sort of awkward questions as does Grothendieck (1984) who questions if the axiomatisation of homological algebra is adequate!)

So the abstraction process continues, clarifies, unifies and leads on.

From a learning point of view, abstraction is a mixed blessing. A learner can often see the abstract well without knowing the motivating examples. We hope to avoid that pitfall. For us however we feel that abstract homotopy helps one understand some of the fundamental structures that occur throughout large areas of mathematics, but which are perhaps clearest in the basic geometric and topological settings. The notion of equivalent maps or processes, where one can be deformed into the other is one such fundamental structure. The topological picture is via a cylinder giving a homotopy between the maps.

2. Cylinders and Cofibrations

Two maps \( f, g : X \to Y \) of topological spaces are homotopic if there exists a map

\[
\phi : X \times I \to Y
\]

such that
\[ \phi(x,0) = f(x), \quad \phi(x,1) = g(x) \quad (x \in X). \]

Here \( X \times I \) denotes the product of \( X \) with the unit interval \( I = [0,1] \) of real numbers. The map \( \phi \) is called a homotopy between \( f \) and \( g \).

For example, any map \( f \) is homotopic to itself using the homotopy
\[ \phi(x,t) = f(x). \]

Hence the basic notion of homotopy theory of topological spaces is induced by the construction of a cylinder
\[ X \times I = X \times [0,1] \]
on a topological space \( X \) together with restriction of functions to the two ends of that cylinder, and the collapsing map sending \((x,t)\) to \(x\).

This leads to the following general definition.

**Definition (2.1).** Let \( C \) be a category. A cylinder, \( I \), on \( C \) is a functor (cylinder functor)
\[ (\ ) \times I : C \rightarrow C \]

\( \) together with three natural transformations
\[ e_0, e_1 : I_{dC} \rightarrow (\ ) \times I, \quad \sigma : (\ ) \times I \rightarrow I_{dC} \]
such that \( \sigma e_0 = \sigma e_1 = I_{dC} \).

If we apply \((\ ) \times I\) to an object \( X \) of \( C \), we shall write simply \( X \times I \), similarly for morphisms. Note that this suggestive notation does not necessarily mean that \( X \times I \) is the product of \( X \) with an object \( I \) of \( C \).

**Examples 1.** The basic example of a cylinder lives in the category \( \text{Top} \) of topological spaces and continuous maps. The construction of the cylinder \( X \times [0,1] \) on a topological space \( X \) together with the maps
\[ e_0(X) : X \rightarrow X \times [0,1] \quad , \quad e_0(X)(x) = (x,0) \]
\[ e_1(X) : X \rightarrow X \times [0,1] \quad , \quad e_1(X)(x) = (x,1) \]
\[ \sigma(X) : X \times [0,1] \rightarrow X \quad , \quad \sigma(X)(x,t) = x \]
determines a canonical cylinder on \( \text{Top} \) which we shall denote by \( T \).

2. For any category \( C \) we have the trivial cylinder \( \text{Id} \) on \( C \) consisting of the identity functor \( I_{dC} : C \rightarrow C \) and the corresponding identity
transformations.

Further examples will be considered in Chapter III.

Given a category $C$ with a cylinder $I = ((\_ \times I, e_0, e_1, \sigma)$, we are in a position to define the basic notions of homotopy theory in $C$. First we define homotopy itself.

**Definition (2.2).** If $f, g : X \to Y$ are morphisms of $C$, then $f$ is homotopic to $g$, written $f \simeq g$, if there is a morphism $\phi : X \times I \to Y$ in $C$ with $\phi e_0(X) = f$, $\phi e_1(X) = g$.

We call $\phi$ a homotopy between $f$ and $g$ and write $\phi : f \simeq g$.

As defined homotopy is neither symmetric nor transitive but is reflexive and compatible with composition.

**Lemma (2.3).** (a) If $f : X \to Y$ is a morphism of $C$, then $f \simeq f$.

(b) Let $h : W \to X$, $f, g : X \to Y$, $k : Y \to Z$ be morphisms of $C$. If $f \simeq g$, then $fh \simeq gh$ and $kf \simeq kg$.

**Proof.** (a) Since $\sigma e_0 = \sigma e_1 = Id$, it follows that $f \sigma(X)$ is a homotopy from $f$ to $f$.

(b) Let $\phi : f \simeq g$ be a homotopy. Then $k \phi$ is a homotopy from $kf$ to $kg$, and $\phi(h \times I)$ is a homotopy from $fh$ to $gh$, since, by naturality of $e_0$,

$$e_0(X)h = (h \times I)e_0(W)$$

and thus

$$fh = \phi e_0(X)h = \phi(h \times I)e_0(W),$$

similarly $gh = \phi(h \times I)e_1(W)$, by naturality of $e_1$. $\square$

We shall see later on what sort of conditions have to be imposed on $(\_ \times I$ to make $\simeq$ into an equivalence relation.

**Definition (2.4).** A morphism $f : X \to Y$ of $C$ is a homotopy equivalence if there is a morphism $g : Y \to X$ of $C$ such that
Such a morphism \( g \) is called a **homotopy inverse** of \( f \).

If \( f : X \to Y, \ g : Y \to X \) are morphisms of \( C \) such that \( fg \simeq Id_Y \), then we say \( g \) is a **right homotopy inverse** of \( f \), and \( f \) is a **left homotopy inverse** of \( g \). Thus a homotopy inverse is both a left and a right homotopy inverse.

**Remark.** Suppose \( \simeq \) is an equivalence relation. We write \( hC \) for \( C/ \simeq \), the quotient category obtained by replacing each \( C(X,Y) \) by \( C(X,Y)/\simeq \). The equivalence class \([f]\) of a morphism \( f \) of \( C \) in \( hC \) will be called its **homotopy class**. Then \( f \) is a homotopy equivalence if and only if \([f]\) is an isomorphism in \( hC \).

**Exercise.** Let \( \pi : C \to hC \) denote the **projection functor**, i.e. \( \pi \) is the identity on objects and maps \( f \in C(X,Y) \) to the homotopy class \([f]\) \( \in hC(X,Y) \).

Show that \( \pi \) satisfies the following universal property:

If \( \rho : C \to D \) is any functor to any category \( D \) such that \( \rho(f) = \rho(g) \) whenever \( f \simeq g \), then there exists a unique functor \( \bar{\rho} : hC \to D \) such that \( \rho = \bar{\rho}\pi \), i.e. the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\rho} & D \\
\pi \downarrow & & \nearrow \bar{\rho} \\
hC & & 
\end{array}
\]

commutes.

There is an important class of morphisms known as cofibrations, definable in any category with cylinder, I. These are the morphisms along which homotopies can be extended. They have a **homotopy extension property** (sometimes abbreviated just as HEP). More precisely we have the following definition.

**Definition (2.5).** (a) A morphism \( i : A \to X \) of \( C \) has the
homotopy extension property (HEP) with respect to an object $Y$ of $C$ if for any pair of morphisms of $C$, $\phi : A \times I \to Y$, $f : X \to Y$, such that $\phi e_0(A) = f i$, there is a morphism $\Phi : X \times I \to Y$ of $C$ such that $\Phi(i \times I) = \phi$ and $\Phi e_0(X) = f$.

(b) A morphism of $C$ is a cofibration if it has the HEP with respect to any object of $C$.

By definition a morphism $i : A \to X$ of $C$ is a cofibration if and only if the diagram

$$
\begin{array}{ccc}
A \times I & \xrightarrow{\phi} & Y \\
\downarrow{e_0(A)} & & \\
A & \xrightarrow{i} & X \\
\downarrow{i} & & \\
X & \xrightarrow{e_0(X)} & X \times I
\end{array}
$$

is a weak pushout in $C$.

We prove a few elementary properties of cofibrations.

**Proposition (2.6).** (a) Any isomorphism is a cofibration.
(b) The composite of cofibrations is a cofibration.

Proof. (a) Let $i : A \to X$ be an isomorphism of $C$. Suppose we are given morphisms $\phi : A \times I \to Y$, $f : X \to Y$ such that $\phi e_0(A) = f i$. Then for $\Phi = \phi(i^{-1} \times I)$ we have

$$
\Phi(i \times I) = \phi(i^{-1} \times I)(i \times I) = \phi(i^{-1}i \times I) = \phi
$$
and, by naturality of $e_0$,
\[
\Phi e_0(X) = \phi (i^{-1} \times I)e_0(X) = \phi e_0(A)i^{-1} = fii^{-1} = f.
\]

(b) Let $i : A \to X$, $i' : X \to X'$ be cofibrations. Suppose we are given morphisms $\phi : A \times I \to Y$, $f' : X' \to Y$ such that
\[
\phi e_0(A) = f'i'i.
\]

We have to show that there is a morphism $\Phi' : X' \times I \to Y$ such that $\Phi' e_0(X') = f'$ and $\Phi'(i'i \times I) = \phi$. Consider the diagram

Since $i$ is a cofibration, there exists $\phi' : X \times I \to Y$ such that $\phi'e_0(X) = f'i'$ and $\phi'(i \times I) = \phi$. Since $i'$ is a cofibration, there exists $\Phi' : X' \times I \to Y$ such that $\Phi' e_0(X') = f'$ and $\Phi'(i' \times I) = \phi'$, hence
\[
\Phi'(i'i \times I) = \Phi'(i' \times I)(i \times I) = \phi'(i \times I) = \phi,
\]

as desired. $\square$

Next we prove that the pushout of a cofibration is again a cofibration provided the cylinder functor is assumed to preserve pushouts.

**Proposition (2.7).** If in the pushout diagram in $C$,
\[ A \xrightarrow{u} C \]
\[ i \downarrow \quad \uparrow j \]
\[ X \xrightarrow{v} Z \]

\[ i \text{ is a cofibration and we also have that } (\_ \times I) \text{ preserves pushouts, then } j \text{ is a cofibration.} \]

**Proof.** Suppose we are given \( \phi : C \times I \to Y, f : Z \to Y \) such that \( \phi e_0(C) = fj \). As \( f \) is a morphism from a pushout, it is uniquely determined by its components \( fj \) and \( fv \). Consider the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & C \\
\downarrow i & & \downarrow j \\
X & \xrightarrow{v} & Z \\
\end{array}
\quad \begin{array}{ccc}
& \xrightarrow{e_0(C)} & \\
& i \downarrow & \uparrow \phi \\
& & j \\
& \downarrow & \\
& X & \xrightarrow{f} Y \\
\end{array}
\]

Since \( e_0 \) is natural, \( e_0(C)u = (u \times I)e_0(A) \) and there is a commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{e_0(A)} & A \times I \\
\downarrow i & & \downarrow \phi(u \times I) \\
X & \xrightarrow{f v} & Y \\
\end{array}
\]

As \( i \) is a cofibration there is a \( \Phi : X \times I \to Y \) satisfying

\[ \Phi(i \times I) = \phi(u \times I) \text{ and } \Phi e_0(X) = fv. \]

We next call on our assumption that \( (\_ \times I) \) preserves pushouts, so

\[
\begin{array}{ccc}
A \times I & \xrightarrow{u \times I} & C \times I \\
\downarrow i \times I & & \downarrow j \times I \\
X \times I & \xrightarrow{v \times I} & Z \times I \\
\end{array}
\]

8
is a pushout. Since \( \phi(u \times I) = \Phi(i \times I) \), the pair \((\phi, \Phi)\) induces a morphism \( \Psi : Z \times I \rightarrow Y \) such that
\[
\Psi(j \times I) = \phi \text{ and } \Psi(v \times I) = \Phi.
\]

We must check that \( \Psi \) is the required homotopy.

Since \( \Psi(j \times I) = \phi \), it remains to verify that \( \Psi e_0(Z) = f \). However
\[
\Psi e_0(Z)j = \Psi(j \times I)e_0(C) = \phi e_0(C) = fj
\]
whilst
\[
\Psi e_0(Z)v = \Psi(v \times I)e_0(X) = \Phi e_0(X) = fv,
\]
and by unicity of \( f \) with these components, we must have \( \Psi e_0(Z) = f \), i.e. \( j \) is a cofibration. \( \square \)

**Remark.** Although the statement of (2.7) asks that the cylinder functor preserves pushouts, the proof only uses that it preserves pushout squares in which one of the basic maps (in this case \( i \)) was a cofibration. This is in fact quite commonly the case for the use of ‘pushout preservation’, but we will not go for the greater generality here. When we consider additive cylinder functors there are examples of cylinders that do not in general preserve all pushouts, but they do preserve those that matter, i.e. those with a cofibration in them as above.

In the topological case, there is a neat construction (the *mapping cylinder* of a morphism) which allows one to factor any morphism as the composite of a cofibration and a homotopy equivalence. Interpreted categorically, this construction adapts well to the abstract setting that we are considering.

Let \( f : X \rightarrow Y \) be a morphism of \( C \) and suppose we have a pushout diagram
\[
\begin{array}{ccc}
X & \overset{e_0(X)}{\longrightarrow} & X \times I \\
\downarrow f & & \downarrow \pi_f \\
Y & \overset{j_f}{\longrightarrow} & M_f
\end{array}
\]

(2.8)
Then $M_f$ is called a **mapping cylinder** of $f$, and we say, $f$ has a **mapping cylinder**. As the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{e_0(X)} & X \times I \\
\downarrow f & & \downarrow f \sigma(X) \\
Y & \xrightarrow{Id_Y} & Y
\end{array}
$$

commutes, there is a morphism $p_f : M_f \to Y$ such that

$$p_f \pi_f = f \sigma(X) \quad \text{and} \quad p_f j_f = Id_Y.
$$

Putting $i_f = \pi_f e_1(X)$, we have a factorisation (**mapping cylinder factorisation**)  

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow i_f & & \downarrow p_f \\
M_f & & \\
\downarrow & & \\
Y & \xrightarrow{e_0(Y)} & Y \times I
\end{array}
$$

(2.9)

of $f$. The properties of the mapping cylinder factorisation will be studied in section 5 of this chapter.

As the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{e_0(X)} & X \times I \\
\downarrow f & & \downarrow f \times I \\
Y & \xrightarrow{e_0(Y)} & Y \times I
\end{array}
$$

commutes, there is a map $s_f : M_f \to Y \times I$ such that $s_f \pi_f = f \times I$ and $s_f j_f = e_0(Y)$.

**Proposition (2.10).** Let $f : X \to Y$ be a morphism of $C$ which has a mapping cylinder $M_f$. Then the following conditions are equivalent.
(a) $f$ is a cofibration.
(b) $f$ has the HEP with respect to $M_f$.
(c) $s_f$ is a section, i.e. there is a morphism $r$ such that $rs_f = Id_{M_f}$.

Proof. (a) $\Rightarrow$ (b) is trivial.
(b) $\Rightarrow$ (c). Suppose $f$ has the HEP with respect to $M_f$. As (2.8) commutes, there exists a morphism $r : Y \times I \rightarrow M_f$ such that $r(f \times I) = \pi_f$ and $re_0(Y) = j_f$. Since

$$rs_f \pi_f = r(f \times I) = \pi_f = Id_{M_f} \pi_f$$
and
$$rs_f j_f = re_0(Y) = j_f = Id_{M_f} j_f,$$
we conclude $rs_f = Id_{M_f}$ by the pushout property of (2.8). Hence $s_f$ is a section.

(c) $\Rightarrow$ (a). Let $r : Y \times I \rightarrow M_f$ be a morphism such that

$rs_f = Id_{M_f}$. Suppose we are given $\phi : X \times I \rightarrow W$, $g : Y \rightarrow W$ such that $\phi e_0(X) = gf$. Since (2.8) is a pushout, there is a morphism $\phi' : M_f \rightarrow W$ such that $\phi' \pi_f = \phi$ and $\phi' j_f = g$. Then $\Phi = \phi' r$ satisfies

$$\Phi(f \times I) = \phi' r(f \times I) = \phi' rs_f \pi_f = \phi' \pi_f = \phi$$
and
$$\Phi e_0(Y) = \phi' re_0(Y) = \phi' rs_f j_f = \phi' j_f = g,$$
i.e. $f$ is a cofibration. $\square$

We conclude our discussion of elementary properties of cofibrations with a property which is related to the problem of extending a map from a subspace to the whole space (see the discussion in tom Dieck–Kamps–Puppe (1970),1.1).

**Proposition (2.11).** Let

$$\begin{array}{ccc}
A & \xrightarrow{i} & Y \\
\downarrow j & & \downarrow f \\
X & \xrightarrow{f} & Y
\end{array}$$

(2.12)

be a diagram in $C$ such that $fi \simeq j$, i.e. (2.12) commutes up to homo-